



Functionals of Complex Ornstein-Uhlenbeck Processes

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Abstract—The exact distribution of the sufficient statistics and distribution of the maximum likelihood estimator of the drift (damping) parameter in a stationary complex Ornstein-Uhlenbeck process, given by (1.1), is investigated. Complete tables of the distribution function for different levels are given by the help of MATLAB. The comparison with the earlier calculations are discussed. The relation of the famous model of Chandler Wobble proposed by Kolmogorov is investigated [1]. © 1998 Elsevier Science Ltd. All rights reserved.

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1. INTRODUCTION

Let us consider the complex-valued stationary autoregressive process $\xi(t) = \xi_1(t) + i\xi_2(t)$, $t \geq 0$, given by the Stochastic Differential Equation (SDE)

$$d\xi(t) = -\gamma\xi(t) dt + \sigma_w dw(t), \quad \xi(0) = \zeta, \quad (1.1)$$

where $w(t) = w_1(t) + iw_2(t)$, $t \geq 0$ is a standard complex Wiener process, i.e., $w_1(t)$ and $w_2(t)$ are independent standard real-valued Wiener processes, $\gamma = \lambda - i\omega$ with $\lambda > 0$, $\omega \in \mathbb{R}$, $\sigma_w > 0$, and $\zeta = \zeta_1 + i\zeta_2$, where ζ_1 and ζ_2 are normal random variables that are chosen according to stationarity. $\xi(t)$ is called a complex Ornstein-Uhlenbeck or complex AR(1) process [2]. Since σ_w can be estimated exactly with probability 1 by observing $\xi(t)$ on an arbitrary time interval, the transformation $\xi'(t) = \xi(t)/\sigma_w$ enables us to investigate only the special case $\sigma_w = 1$.

We are interested in the estimation of the drift (damping) parameter λ and period ω in the case when the process is observed on an interval of the form $[0, T]$, $T > 0$. Our main goal is to reproduce with more precise confidence limits by the help of PC, and MATLAB, the table of the distribution of the maximum likelihood estimator of the drift parameter λ and compute that of the sufficient statistics for λ , given in [2,3].

The stochastic process defined by (1.1) can also be interpreted in the following two-dimensional form. Consider the two-dimensional real-valued stationary autoregressive process $(\xi_1(t), \xi_2(t))^T$, $t \geq 0$, given by the SDE

$$\begin{pmatrix} d\xi_1(t) \\ d\xi_2(t) \end{pmatrix} = \begin{pmatrix} -\lambda & -\omega \\ \omega & -\lambda \end{pmatrix} \begin{pmatrix} \xi_1(t) dt \\ \xi_2(t) dt \end{pmatrix} + \begin{pmatrix} dw_1(t) \\ dw_2(t) \end{pmatrix}. \quad (1.2)$$

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Then, $(\xi_1(t), \xi_2(t))^T$ is a two-dimensional real-valued Ornstein-Uhlenbeck process and the complex process $\xi_1(t) + i\xi_2(t)$ is a complex Ornstein-Uhlenbeck process. Conversely, if $\xi_1(t)$ and $\xi_2(t)$ are the real and imaginary parts of a complex Ornstein-Uhlenbeck process defined by (1.1), then $(\xi_1(t), \xi_2(t))^T$ satisfies the SDE (1.2). Introduce the notation

$$A = \begin{pmatrix} -\lambda & -\omega \\ \omega & -\lambda \end{pmatrix}.$$

Then, the covariance matrix function of the process $(\xi_1(t), \xi_2(t))^T$ is given by

$$R(\tau) = \mathbb{E} \begin{pmatrix} \xi_1(t+\tau) \\ \xi_2(t+\tau) \end{pmatrix} (\xi_1(t), \xi_2(t)) = e^{A\tau} R(0) = e^{-\lambda\tau} \begin{pmatrix} \cos \tau\omega & -\sin \tau\omega \\ \sin \tau\omega & \cos \tau\omega \end{pmatrix} R(0), \quad \tau \geq 0,$$

where $R(0) = (1/2\lambda)I$. Hence, the complex covariance function of the complex Ornstein-Uhlenbeck process defined by (1.1) is given by

$$C(\tau) = \mathbb{E}\xi(t+\tau)\overline{\xi(t)} = \frac{1}{\lambda}e^{-\lambda\tau}(\cos \omega\tau + i\sin \omega\tau), \quad \tau \geq 0,$$

i.e., it behaves like a damped oscillation with frequency ω .

The discretised process with stepsize $\Delta > 0$ defined by $X_k = \xi(k\Delta)$, $k \in \mathbb{N}$, satisfies the following stochastic difference equation:

$$X_k = e^{A\Delta} X_{k-1} + \varepsilon_k = Q X_{k-1} + \varepsilon_k, \quad k \in \mathbb{N},$$

where the ε_k 's are two-dimensional Gaussian i.i.d. random variables with zero mean and covariance matrix R_ε . We note that this two-dimensional autoregressive model is asymptotically stationary, since all the eigenvalues of the matrix $Q = e^{A\Delta}$ lie inside the unit circle (the real parts of the two eigenvalues of A given by $-\lambda \pm i\omega$ are negative). Then, the covariance matrix function is $R(\tau) = e^{Q\tau} R(0)$, where $R(0)Q + QR(0) = R_\varepsilon$.

An application of the complex Ornstein-Uhlenbeck process is the description of the motion of the instantaneous axis of the Earth's rotation. This motion has a one-year period and if it is removed, there remains the so-called Chandler Wobble, which has a period about 435 days (14 months). The Chandler Wobble with estimation of the period was investigated by several authors (see, e.g., [4–11]). Kolmogorov proposed the complex stochastic process

$$z(t) = x(t) + iy(t) = me^{i2\pi t} + \xi(t)$$

to describe the Chandler Wobble, i.e., the motion of the pole, where $x(t)$ and $y(t)$ are the coordinates of the deviation of the instantaneous pole from the North Pole (see [1]). In that model the first term is a periodical component, and the second term $\xi(t)$ is a complex Ornstein-Uhlenbeck process.

In the two-dimensional first-order autoregressive AR(1) model with discrete time, the observations X_k at the time k are generated by the following stochastic difference equation:

$$\begin{aligned} X_k &= Q X_{k-1} + \varepsilon_k, \quad k = 1, 2, \dots, \\ X_0 &= 0, \end{aligned} \tag{1.3}$$

where the ε_k 's are two-dimensional random disturbances or innovations, Q is an unknown real parameter matrix. The main goal of the statistical theory of the autoregressive processes is to construct estimators of the parameters based on the observations X_1, X_2, \dots, X_n , $n \in \mathbb{N}$, and to characterize their exact or asymptotic distributions. The Least Squares Estimator (LSE) of the parameter matrix Q of the process defined by (1.3) is given by

$$\hat{Q} = \left(\sum_{k=1}^n X_k X_{k-1}^T \right) \left(\sum_{k=1}^n X_{k-1} X_{k-1}^T \right)^{-1}. \tag{1.4}$$

It is well known that if the model is asymptotically stationary, i.e., all the eigenvalues of the parameter matrix Q lie inside the unit circle, then the standardized LSE of Q is asymptotically normal in case the innovations are independent and identically distributed, see, e.g., [12,13]. Previously, White [14], Arató [15] and others pointed out that for the first-order one-dimensional autoregressive process in situations where the parameter is close to one, i.e., in the nearly unstable case, the asymptotic distribution of the standardized LSE of the parameter do not follow the standard normal law even in the cases when the innovations are independent standard normal random variables. Recently, Chan and Wei [16] and Pap and van Zuijlen [17] proved that the situation is similar in the case of higher-order or multidimensional autoregressive processes. In fact, they showed that if the eigenvalues of the parameter matrix Q lie on the complex unit circle, the limit law of the standardized LSE of the parameters is a functional of a multidimensional Wiener process and it does not follow the normal law. More precisely, let us consider the so-called nearly unstable two-dimensional VAR(1) model, where Q in model (1.3) is replaced by $Q_n = \exp\{(C_n/n + B)\}$, where C_n , $n \geq 1$, B are 2×2 real matrices such that $C_n \rightarrow C$, B is a known skew-symmetric matrix, and $C_n B = B C_n$, $n \geq 1$. Then, the nearly instability holds, since $Q_n \rightarrow \exp\{B\}$ and $\exp\{B\}$ is an orthogonal matrix. In that case, the LSE \widehat{Q}_n of the parameter matrix Q_n is given by (1.4), too. Then,

$$n(\widehat{Q}_n - Q_n) \Rightarrow \int_0^1 (dW(t))Y^\top(t) \left(\int_0^1 Y(t)Y^\top(t) dt \right)^{-1},$$

where the process $Y(t)$, $t \in [0, 1]$ is given by

$$dY(t) = CY(t)dt + dW(t), \quad Y(0) = 0$$

and $W(t)$, $t \in [0, 1]$ is a standard two-dimensional Wiener process.

The relation between the discrete and continuous time multidimensional autoregressive processes and their parameter estimations were discussed also in [17,18].

2. MAXIMUM LIKELIHOOD ESTIMATION OF THE PARAMETERS

Consider again the complex Ornstein-Uhlenbeck process (1.1) with the drift (or damping) parameter λ and period ω . It is well known that the two-dimensional process $X(t) = (\xi_1(t), \xi_2(t))^\top$ defined by (1.2) is a stationary Gauss-Markov process and

$$X(t) = X(0) + \int_0^t e^{(t-s)A} dW(s),$$

where $W(t)$, $t \geq 0$ is a standard two-dimensional Wiener process. Denote by f the absolutely continuous density function of the random initial value $X(0) = \zeta$. Let $\mathbb{P}_{T,X}$ be the measure generated on $C^2[0, T]$ by the process $X(s)$, $0 \leq s \leq T$. Let us define the measure $\mathbb{P}_{T,W}$ on $C^2[0, T]$ as the product of the two-dimensional Lebesgue measure and the conditional measure generated by the process $W(s)$, $0 \leq s \leq T$, on $C_0^2[0, T]$. (We note that $C_0^2[0, T] := \{x \in C^2[0, T] \mid x(0) = 0\}$ and $C^2[0, T] = \mathbb{R}^2 \times C_0^2[0, T]$.) Then, the measure $\mathbb{P}_{T,X}$ is absolutely continuous with respect to $\mathbb{P}_{T,W}$ and its Radon-Nikodym derivative has the following form:

$$\frac{d\mathbb{P}_{T,X}}{d\mathbb{P}_{T,W}}(X) = f(X(0)) \exp \left\{ \int_0^T \langle AX(u), dX(u) \rangle - \frac{1}{2} \int_0^T |AX(u)|^2 du \right\} \quad (2.5)$$

(see, e.g., [2]). Elementary calculation shows that if we start the process according to stationarity, then

$$\frac{d\mathbb{P}_{T,X}}{d\mathbb{P}_{T,W}}(X) = \frac{\lambda}{\pi} \exp \left\{ -\frac{\lambda^2 + \omega^2}{2} T s_2^2 - \lambda s_1^2 + \lambda T + \omega T r \right\}, \quad (2.6)$$

where

$$s_1^2 = \frac{1}{2} (|X(0)|^2 + |X(T)|^2), \quad s_2^2 = \frac{1}{T} \int_0^T |X(u)|^2 du,$$

$$r = \frac{1}{T} \int_0^T |X(u)|^2 d\theta(u) = \frac{1}{T} \int_0^T (\xi_1(u) d\xi_2(u) - \xi_2(u) d\xi_1(u)),$$

and $\theta(t) = \arctan(\xi_2(t)/\xi_1(t))$, $t \geq 0$. We note that r is the so-called Levy's random area (see [19]). It is easy to see, that

$$\mathbb{E}s_1^2 = \frac{1}{\lambda} \quad \text{and} \quad \mathbb{E}s_2^2 = \frac{1}{T} \int_0^T \mathbb{E}|X(t)|^2 dt = \frac{1}{\lambda}.$$

This implies that besides the maximum likelihood estimator we have two other estimators of λ , namely, $1/s_1^2$ and $1/s_2^2$.

From (2.6), we get that the maximum likelihood estimators of the parameters λ and ω (the unique positive solutions of the likelihood equation) are

$$\hat{\lambda} = \frac{-(s_1^2 - T) + \sqrt{(s_1^2 - T)^2 + 4Ts_2^2}}{2Ts_2^2}, \quad (2.7)$$

$$\hat{\omega} = \frac{r}{s_2^2}. \quad (2.8)$$

It is well known that the distribution of $(\hat{\omega} - \omega)\sqrt{Ts_2^2}$ is exactly, and not asymptotically, standard normal (see [1,19]) and it does not depend on λ . A generalization of this fact for multidimensional autoregressive processes can be found in [20]. On the basis of this observation, it is easy to construct confidence intervals for the unknown period parameter ω . In order to give confidence intervals for the drift parameter λ , we have to determine the exact probability distribution of the estimator $\hat{\lambda}$. In the following theorem, we give an explicite formula for the joint characteristic function Ψ of the sufficient statistics s_1^2 and Ts_2^2 .

THEOREM 2.1. (See [2, Section 4.2].) *Let us introduce the notation $\Lambda = \sqrt{\lambda^2 - 2i\alpha_2}$; then*

$$\begin{aligned} \Psi(\alpha_1, \alpha_2) &= \mathbb{E} \exp(i\alpha_1 s_1^2 + i\alpha_2 Ts_2^2) \\ &= \frac{4\lambda\Lambda \exp(\lambda T)}{(\lambda - i\alpha_1 + \Lambda)^2 \exp(\Lambda T) - (\lambda - i\alpha_1 - \Lambda)^2 \exp(-\Lambda T)}. \end{aligned} \quad (2.9)$$

Hence, the characteristic functions of the marginal distributions are given by

$$\Psi_{s_1^2}(\alpha) = \Psi(\alpha, 0) = \frac{4\lambda^2 \exp(\lambda T)}{(2\lambda - i\alpha)^2 \exp(\lambda T) + \alpha^2 \exp(-\lambda T)} \quad (2.10)$$

and

$$\Psi_{Ts_2^2}(\alpha) = \Psi(0, \alpha) = \frac{4\lambda\Lambda \exp(\lambda T)}{(\lambda + \Lambda)^2 \exp(\Lambda T) - (\lambda - \Lambda)^2 \exp(-\Lambda T)}. \quad (2.11)$$

Since

$$\Psi_{s_1^2}(\alpha) = i\lambda \exp(\lambda T) \left(\left(\alpha + \frac{2i\lambda}{1 + \exp(-\lambda T)} \right)^{-1} - \left(\alpha + \frac{2i\lambda}{1 - \exp(-\lambda T)} \right)^{-1} \right),$$

by Levy's inversion formula (see [21, Theorem 3.3.2]) and formula VII. In Section 2, (4) of [22], we have

$$\begin{aligned} f_{s_1^2}(t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \Psi_{s_1^2}(\alpha) \exp(-i\alpha t) d\alpha \\ &= \begin{cases} \lambda e^{\lambda T} \left(\exp\left(-\frac{2\lambda t}{1 + \exp(-\lambda T)}\right) - \exp\left(-\frac{2\lambda t}{1 - \exp(-\lambda T)}\right) \right), & \text{if } t \geq 0, \\ 0, & \text{otherwise.} \end{cases} \end{aligned} \quad (2.12)$$

Hence, the distribution function of s_1^2 is the following:

$$F_{s_1^2}(x) = \begin{cases} \frac{e^{\lambda T} - 1}{2} \exp\left(-\frac{2\lambda x}{1 - e^{-\lambda T}}\right) - \frac{e^{\lambda T} + 1}{2} \exp\left(-\frac{2\lambda x}{1 + e^{-\lambda T}}\right) + 1, & \text{if } x \geq 0, \\ 0, & \text{otherwise.} \end{cases} \quad (2.13)$$

In Table 1, the quantiles $z_p = z_p(\lambda)$ of $1/s_1^2$ are given for $T = 1$, i.e.,

$$\mathbb{P}_\lambda \left(\frac{1}{s_1^2} > z_p \right) = p.$$

The distribution function is tabulated for $p = 0.001, 0.01, 0.025, 0.05, 0.1, 0.95, 0.975, 0.99, 0.999$ and for $\lambda = 0.001, 0.01, 0.05, 0.1-1$ (by stepsize 0.1), $1-5$ (by stepsize 0.5).

Table 1. The quantiles z_p of $1/s_1^2$ for $T = 1$.

$\lambda \setminus p$	0.100	0.050	0.025	0.010	0.001	0.999	0.990	0.975	0.950	0.900
0.001	0.0095	0.0193	0.0388	0.0948	0.6786	1.4×10^{-4}	2.2×10^{-4}	2.7×10^{-4}	3.3×10^{-4}	4.3×10^{-4}
0.01	0.0911	0.1785	0.3315	0.6791	2.8420	0.0015	0.0022	0.0027	0.0033	0.0044
0.05	0.3927	0.6906	1.1008	1.9415	6.8978	0.0074	0.0111	0.0138	0.0170	0.0220
0.1	0.6921	1.1149	1.7301	2.9516	10.1353	0.0151	0.0225	0.0281	0.0345	0.0446
0.2	1.1412	1.7897	2.7047	4.5175	15.1718	0.0314	0.0467	0.0580	0.0709	0.0914
0.3	1.5358	2.3689	3.5427	5.8675	19.5287	0.0488	0.0723	0.0895	0.1092	0.1399
0.4	1.9032	2.9092	4.3262	7.1322	23.6199	0.0672	0.0993	0.1225	0.1489	0.1899
0.5	2.2576	3.4316	5.0849	8.3587	27.5948	0.0866	0.1274	0.1568	0.1900	0.2409
0.6	2.6057	3.9458	5.8329	9.5692	31.5236	0.1068	0.1565	0.1921	0.2320	0.2928
0.7	2.9514	4.4572	6.5775	10.7757	35.4437	0.1278	0.1865	0.2282	0.2747	0.3451
0.8	3.2966	4.9687	7.3231	11.9848	39.3759	0.1495	0.2172	0.2650	0.3179	0.3976
0.9	3.6427	5.4821	8.0722	13.2003	43.3321	0.1716	0.2484	0.3021	0.3615	0.4503
1.0	3.9904	5.9985	8.8260	14.4243	47.3189	0.1942	0.2799	0.3396	0.4051	0.5030

Unfortunately, the distribution function of the statistics Ts_2^2 cannot be given in a closed form, but the following lemma is quite useful in numerical computations. We note that if the characteristic function φ of a nonnegative random variable is absolutely integrable, the corresponding distribution function is given by

$$F(x) = \begin{cases} \frac{1}{2\pi} \int_{-\infty}^{\infty} \varphi(\alpha) \frac{1 - \exp(-i\alpha x)}{i\alpha} d\alpha, & \text{if } x \geq 0, \\ 0, & \text{otherwise.} \end{cases} \quad (2.14)$$

LEMMA 2.2.

- (i) The characteristic function $\Psi_{Ts_2^2}(\alpha)$ tends to 0 as $\alpha \rightarrow \infty$ in exponential order. More precisely,

$$\Psi_{Ts_2^2}(\alpha) = o\left(\sinh^{-1}\left(\sqrt{\lambda^2 - 2i\alpha}\right)\right),$$

where the notation $f(\alpha) = o(g(\alpha))$ means that $\lim_{\alpha \rightarrow \infty} |f(\alpha)/g(\alpha)| = 0$.

- (ii) The real part of $\Psi_{Ts_2^2}(\alpha)$ is a symmetric, the imaginary part is an antisymmetric function of α .
 (iii) The function

$$h_{Ts_2^2}(\alpha) = \frac{1}{\alpha} \left(\mathcal{R} \left(\Psi_{Ts_2^2}(\alpha) \right) \sin(\alpha x) + \mathcal{I} \left(\Psi_{Ts_2^2}(\alpha) \right) \right) (1 - \cos(\alpha x)), \quad (2.15)$$

equals zero at the points $2\pi k/x$, $k \in \mathbb{Z} \setminus \{0\}$, and has at most one root in the intervals of form $(2\pi k/x, 2\pi(k+1)/x)$.

PROOF. (i) Elementary calculations show that the characteristic function $\Psi_{Ts_2^2}(\alpha)$ can be rewritten into the following form:

$$\Psi_{Ts_2^2}(\alpha) = \frac{\exp(\lambda T)}{\sinh(\Lambda T)(\coth(\Lambda T) + (\lambda^2 + \Lambda^2/2\lambda\Lambda))}.$$

Take the trigonometric form of $\Lambda = \varrho(\cos \varphi + i \sin \varphi)$. Then, the assertion follows from the limits

$$\begin{aligned} \left| \frac{\lambda^2 + \Lambda^2}{2\lambda\Lambda} \right|^2 &= \frac{\lambda^2}{4\varrho^2} + \frac{1}{2} (\cos^2 \varphi - \sin^2 \varphi) + \frac{\varrho^2}{4\lambda^2} \longrightarrow \infty, \\ |\coth(\Lambda T)|^2 &= \frac{\cosh(2\varrho T \cos \varphi) + \cos(2\varrho T \sin \varphi)}{\cosh(2\varrho T \cos \varphi) - \cos(2\varrho T \sin \varphi)} \longrightarrow 1, \end{aligned}$$

if $\alpha \rightarrow \infty$ implying $\varrho \rightarrow \infty$ and $\varphi \rightarrow -\pi/4$.

(ii) This statement can be proved by long but straightforward calculations.

(iii) The roots of the equation $h_{Ts_2^2}(\alpha) = 0$ satisfy one of the following two equations:

$$\cos(\alpha x) = 1, \quad \frac{\mathcal{R}(\Psi_{Ts_2^2}(\alpha))}{\mathcal{I}(\Psi_{Ts_2^2}(\alpha))} = \frac{\cos(\alpha x) - 1}{\sin(\alpha x)} = \tan\left(\frac{\alpha x}{2}\right).$$

By the first equation, we have roots $2\pi k/x$, $k \in \mathbb{Z} \setminus \{0\}$. It can be proved that the left- and right-hand sides of the second equation are monotone functions of α in the intervals of form $(2\pi k/x, (2\pi(k+1)/x))$, $k \in \mathbb{Z}$. Hence, $h_{Ts_2^2}(\alpha)$ has at most one root in the intervals of the above form.

Lemma 2.2 and (2.14) imply that the distribution function of Ts_2^2 is the following:

$$F_{Ts_2^2}(x) = \begin{cases} \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{\alpha} \left(\mathcal{R}(\Psi_{Ts_2^2}(\alpha)) \sin(\alpha x) \right. \\ \quad \left. + \mathcal{I}(\Psi_{Ts_2^2}(\alpha)) (1 - \cos(\alpha x)) \right) d\alpha, & \text{if } x \geq 0, \\ 0, & \text{otherwise.} \end{cases} \quad (2.16)$$

In Table 2, the quantiles $z_p = z_p(\lambda)$ of $1/s_2^2$ are given for $T = 1$, i.e.,

$$\mathbb{P}_\lambda \left(\frac{1}{s_2^2} > z_p \right) = p.$$

The distribution function is tabulated for $p = 0.001, 0.01, 0.025, 0.05, 0.1, 0.95, 0.975, 0.99, 0.999$ and for $\lambda = 0.05, 0.1-1$ (by stepsize 0.1), $1-10$ (by stepsize 0.5), $10-100$ (by stepsize 10).

According to Arató [3], to determine the distribution of the maximum likelihood estimator $\hat{\lambda}$ of λ given by (2.7), let us consider the following equation:

$$\mathbb{P}_\lambda \left(\hat{\lambda} < x \right) = \mathbb{P}_\lambda \left(x^2 s_2^2 + x s_1^2 > Tx + 1 \right). \quad (2.17)$$

Denote $\lambda T = \kappa$, $\hat{\lambda} T = \hat{\kappa}$, and $\zeta_y = \lambda y s_1^2 + \lambda^2 y^2 s_2^2$. Using this notation, (2.17) can be rewritten as follows:

$$\mathbb{P}_\kappa \left(\hat{\kappa} < \kappa y \right) = \mathbb{P}_\kappa (\zeta_y > \kappa y + 1). \quad (2.18)$$

Theorem 2.1 implies that the characteristic function of ζ_y is

$$\Psi_{\zeta_y}(\alpha) = \frac{4\Lambda_y \exp(\kappa)}{(1 - iy\alpha + \Lambda_y)^2 \exp(\kappa\Lambda_y) - (1 - iy\alpha - \Lambda_y)^2 \exp(-\kappa\Lambda_y)}, \quad (2.19)$$

where $\Lambda_y = \sqrt{1 - 2iy^2\alpha}$.

The following lemma is analogous to Lemma 2.2 and can be proved in a similar way.

Table 2. The quantiles z_p of $1/s_2^2$ for $T = 1$.

$\lambda \setminus p$	0.100	0.050	0.025	0.010	0.001	0.999	0.990	0.975	0.950	0.900
0.05	0.4162	0.7471	1.2174	2.0261	4.7293	0.0073	0.0110	0.0137	0.0169	0.0219
0.1	0.7431	1.2246	1.8296	2.7832	5.7264	0.0149	0.0223	0.0278	0.0341	0.0442
0.2	1.2378	1.8649	2.5913	3.6743	6.8315	0.0306	0.0457	0.0568	0.0697	0.0900
0.3	1.6169	2.3256	3.1194	4.2738	7.5502	0.0472	0.0702	0.0872	0.1067	0.1374
0.4	1.9338	2.6999	3.5409	4.7454	8.1061	0.0645	0.0958	0.1187	0.1450	0.1861
0.5	2.2119	3.0229	3.9007	5.1443	8.5715	0.0827	0.1225	0.1516	0.1847	0.2363
0.6	2.4637	3.3119	4.2203	5.4963	8.9789	0.1017	0.1503	0.1856	0.2256	0.2877
0.7	2.6964	3.5768	4.5114	5.8154	9.3461	0.1216	0.1791	0.2207	0.2678	0.3404
0.8	2.9147	3.8234	4.7812	6.1100	9.6836	0.1422	0.2090	0.2570	0.3112	0.3943
0.9	3.1217	4.0560	5.0347	6.3858	9.9982	0.1637	0.2398	0.2945	0.3558	0.4493
1.0	3.3196	4.2772	5.2751	6.6467	10.2948	0.1858	0.2717	0.3330	0.4014	0.5053
1.5	4.2176	5.2707	6.3469	7.8027	11.5997	0.3086	0.4452	0.5405	0.6449	0.7992
2.0	5.0242	6.1521	7.2898	8.8122	12.7290	0.4497	0.6403	0.7702	0.9098	1.1116
2.5	5.7791	6.9703	8.1603	9.7397	13.7600	0.6080	0.8541	1.0182	1.1918	1.4380
3.0	6.5006	7.7477	8.9839	10.6139	14.7272	0.7815	1.0838	1.2815	1.4876	1.7754
3.5	7.1987	8.4964	9.7745	11.4506	15.6493	0.9688	1.3273	1.5576	1.7949	2.1223
4.0	7.8793	9.2235	10.5404	12.2591	16.5376	1.1685	1.5825	1.8445	2.1118	2.4769
4.5	8.5463	9.9338	11.2869	13.0456	17.3993	1.3792	1.8480	2.1409	2.4371	2.8382
5.0	9.2023	10.6306	12.0178	13.8144	18.2394	1.5997	2.1226	2.4455	2.7697	3.2055
5.5	9.8492	11.3162	12.7358	14.5683	19.0617	1.8292	2.4052	2.7574	3.1087	3.5780
6.0	10.4885	11.9924	13.4428	15.3098	19.8688	2.0668	2.6951	3.0759	3.4536	3.9552
6.5	11.1213	12.6604	14.1405	16.0405	20.6628	2.3106	2.9915	3.4003	3.8036	4.3366
7.0	11.7484	13.3214	14.8300	16.7618	21.4454	2.5634	3.2940	3.7300	4.1584	4.7218
7.5	12.3405	13.9762	15.5123	17.4749	22.2178	2.8213	3.6018	4.0647	4.5174	5.1105
8.0	12.9882	14.6255	16.1881	18.1806	22.9812	3.0850	3.9148	4.4038	4.8804	5.5023
8.5	13.6019	15.2698	16.8582	18.8797	23.7365	3.3541	4.2324	4.7471	5.2470	5.8971
9.0	14.2120	15.9096	17.5231	19.5728	24.4845	3.6281	4.5543	5.0942	5.6170	6.2946
9.5	14.8188	16.5453	18.1833	20.2604	25.2257	3.9069	4.8802	5.4450	5.9901	6.6946
10	15.4227	17.1773	18.8391	20.9430	25.9607	4.1901	5.2099	5.7990	6.3661	7.0969
20	27.1066	29.3231	31.3777	33.9276	39.8299	10.5068	12.3487	13.3582	14.2990	15.4731
30	38.3917	40.9650	43.3252	46.2248	52.8326	17.5720	20.0953	21.4451	22.6845	24.2087
40	49.4742	52.3497	54.9696	58.1675	65.3821	25.0573	28.1750	29.8182	31.3136	33.1367
50	60.4279	63.5706	66.4204	69.8831	77.6388	32.8218	36.4731	38.3780	40.1011	42.1892
60	71.2903	74.6750	77.7334	81.4367	89.6859	40.7895	44.9286	47.0721	49.0022	51.3310
70	82.0835	85.6912	88.9418	92.8673	101.5730	48.9125	53.5046	55.8687	57.9901	60.5408
80	92.8219	96.6374	100.0675	104.2001	113.3326	57.1611	62.1767	64.7469	67.0469	69.8047
90	103.5156	107.5265	111.1252	115.4530	124.9876	65.5119	70.9277	73.6921	76.1602	79.1127
100	114.1719	118.3676	122.1261	126.6387	136.5551	73.9495	79.7450	82.6937	85.3209	88.4579

LEMMA 2.3.

(i) The characteristic function $\Psi_{\zeta_v}(\alpha)$ tends to 0 as $\alpha \rightarrow \infty$ in exponential order, that is,

$$\Psi_{\zeta_v}(\alpha) = o\left(\sinh^{-1}\left(\kappa\sqrt{1-2iy^2\alpha}\right)\right).$$

(ii) The real part of $\Psi_{\zeta_v}(\alpha)$ is a symmetric, the imaginary part is an antisymmetric function of α .

(iii) The function

$$h_{\zeta_v}(\alpha) = \frac{1}{\alpha}(\mathcal{R}(\Psi_{\zeta_v}(\alpha))\sin(\alpha x) + \mathcal{I}(\Psi_{\zeta_v}(\alpha))(1 - \cos(\alpha x))) \quad (2.20)$$

equals zero at the points $2\pi k/x$, $k \in \mathbb{Z} \setminus \{0\}$, and has at most one root in the intervals of form $(2\pi k/x, (2\pi(k+1)/x))$.

Table 3. The quantiles z_p of $\hat{\kappa}$.

$\kappa \setminus p$	0.100	0.050	0.025	0.010	0.001	0.999	0.990	0.975	0.950	0.900
0.05	0.1039	0.1511	0.2161	0.3401	0.9573					
0.1	0.2073	0.2989	0.4222	0.6481	1.6495				0.0300	0.0361
0.2	0.4113	0.5825	0.8035	1.1856	2.6601				0.0615	0.0738
0.3	0.6105	0.8508	1.1508	1.6485	3.4193	0.0473	0.0677	0.0808	0.0943	0.1128
0.4	0.8049	1.1059	1.4717	2.0599	4.0427	0.0652	0.0964	0.1103	0.1283	0.1531
0.5	0.9949	1.3497	1.7716	2.4336	4.5801	0.0839	0.1187	0.1409	0.1635	0.1947
0.6	1.1808	1.5839	2.0544	2.7786	5.0579	0.1036	0.1457	0.1725	0.1997	0.2343
0.7	1.3629	1.8097	2.3231	3.1009	5.4921	0.1240	0.1737	0.2050	0.2369	0.2811
0.8	1.5416	2.0282	2.5800	3.4049	5.8926	0.1452	0.2025	0.2385	0.2751	0.3258
0.9	1.7171	2.2404	2.8269	3.6938	6.2663	0.1672	0.2322	0.2728	0.3142	0.3715
1.0	1.8897	2.4469	3.0650	3.9610	6.6183	0.1898	0.2625	0.3080	0.3542	0.4182
1.5	2.7165	3.4136	4.1591	5.2134	8.1553	0.3119	0.4249	0.4949	0.5662	0.6650
2.0	3.4973	4.3027	5.1436	6.3080	9.4628	0.4469	0.6030	0.6993	0.7971	0.9321
2.5	4.2453	5.1396	6.0582	7.3112	10.6368	0.5938	0.7960	0.9199	1.0452	1.2170
3.0	4.9687	5.9395	6.9242	8.2523	11.7224	0.7525	1.0032	1.1555	1.3087	1.5172
3.5	5.6730	6.7112	7.7540	9.1479	12.7448	0.9230	1.2237	1.4048	1.5859	1.8307
4.0	6.3618	7.4609	8.5558	10.0087	13.7196	1.1051	1.4565	1.6665	1.8752	2.1556
4.5	7.0381	8.1927	9.3353	10.8420	14.6571	1.2981	1.7006	1.9392	2.1752	2.4903
5.0	7.7038	8.9098	10.0963	11.6528	15.5644	1.5017	1.9549	2.2218	2.4846	2.8336
5.5	8.3605	9.6145	10.8420	12.4448	16.4467	1.7149	2.2185	2.5133	2.8023	3.1845
6.0	9.0093	10.3085	11.5746	13.2210	17.3079	1.9370	2.4905	2.8127	3.1276	3.5420
6.5	9.6514	10.9933	12.2958	13.9835	18.1510	2.1674	2.7702	3.1195	3.4596	3.9054
7.0	10.2875	11.6700	13.0071	14.7341	18.9784	2.4055	3.0568	3.4327	3.7976	4.2741
7.5	10.9182	12.3395	13.7097	15.4741	19.7921	2.6506	3.3499	3.7520	4.1411	4.6476
8.0	11.5442	13.0026	14.4045	16.2049	20.5935	2.9023	3.6489	4.0767	4.4896	5.0255
8.5	12.1658	13.6599	15.0923	16.9273	21.3840	3.1601	3.9534	4.4065	4.8427	5.4072
9.0	12.7835	14.3120	15.7738	17.6421	22.1647	3.4235	4.2629	4.7409	5.2001	5.7926
9.5	13.3976	14.9593	16.4495	18.3502	22.9364	3.6920	4.5771	5.0797	5.5613	6.1813
10	14.0084	15.6023	17.1200	19.0519	23.7000	3.9655	4.8956	5.4224	5.9261	6.5730
20	25.7861	27.8890	29.8424	32.2717	37.9144	10.1478	11.8876	12.8368	13.7201	14.8220
30	37.1214	39.6026	41.8810	44.6832	51.0817	17.1326	19.5553	20.8491	22.0366	23.4970
40	48.2361	51.0325	53.5821	56.6965	63.7319	24.5623	27.5834	29.1746	30.6225	32.3878
50	59.2128	62.2850	65.0724	68.4610	76.0578	32.2855	35.8443	37.7006	39.3796	41.4146
60	70.0925	73.4132	76.4150	80.0511	88.1565	40.2200	44.2715	46.3691	48.2580	50.5372
70	80.8994	84.4481	87.6466	91.5102	100.0836	48.3169	52.8248	55.1454	57.2279	59.7320
80	91.6491	95.4096	98.7910	102.8662	111.8756	56.5434	61.4783	64.0071	66.2700	68.9837
90	102.3521	106.3113	109.8644	114.1382	123.5574	64.8755	70.2136	72.9384	75.3711	78.2817
100	113.0163	117.1632	120.8786	125.3403	135.1477	73.2969	79.0175	81.9280	84.5215	87.6183

For the distribution function of ζ_y , we get a formula, similar to (2.16)

$$F_{\zeta_y}(x) = \begin{cases} \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{\alpha} (\mathcal{R}(\Psi_{\zeta_y}(\alpha)) \sin(\alpha x) \\ \quad + \mathcal{I}(\Psi_{\zeta_y}(\alpha))) (1 - \cos(\alpha x)) d\alpha, & \text{if } x \geq 0, \\ 0, & \text{otherwise.} \end{cases} \quad (2.21)$$

In Table 3, the quantiles $z_p = z_p(\kappa)$ of $\hat{\kappa}$ are given, that is,

$$\mathbb{P}_{\kappa}(\hat{\kappa} > z_p) = p.$$

The distribution function is tabulated for $p = 0.001, 0.01, 0.025, 0.05, 0.1, 0.95, 0.975, 0.99, 0.999$ and for $\kappa = 0.05, 0.1-1$ (by stepsize 0.1), $1-10$ (by stepsize 0.5), $10-100$ (by stepsize 10).

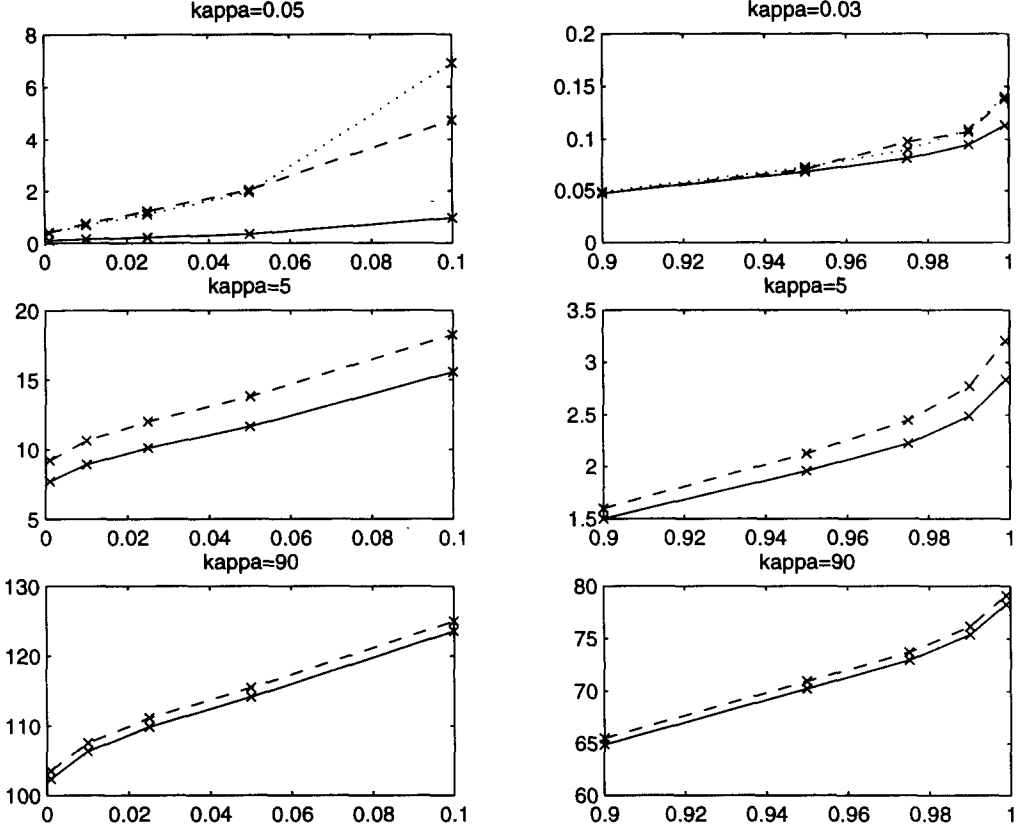


Figure 1. The quantiles for different estimators of $\hat{\kappa} = \hat{\lambda}T$ for $T = 1$ (solid line: the maximum likelihood estimator; dashed line: $1/s_2^2$; dotted line: $1/s_1^2$).

For the small values of κ ($\kappa \leq 0.5$), instead of the maximum likelihood estimator we can use either the estimator $1/s_1^2$ or the estimator $1/s_2^2$. This approximation is very accurate for large values of p . For the small values of p , the quantiles of $1/s_1^2$ and $1/s_2^2$ are close to each other but they do not follow that ones of the maximum likelihood estimator. For the medium values of κ ($1 < \kappa < 50$), the confidence intervals of $1/s_2^2$ are longer than the confidence intervals of the maximum likelihood estimator. For the large κ values ($\kappa > 50$), the two estimators are very close to each other, so for computational reasons it is easier to use the estimator $1/s_2^2$. Figure 1 shows the behaviour of the quantiles of the three estimators for some values of κ .

For small or large values of κ , the following asymptotics can be used for computing the confidence limits.

THEOREM 2.4. *If $\kappa \rightarrow 0$, then,*

$$\lim_{\kappa \rightarrow 0} \mathbb{P}_\kappa(\hat{\kappa} < y\kappa) = \exp\left\{-\frac{1}{y}\right\},$$

i.e., the ratio $\hat{\kappa}/\kappa$ has a χ^2 distribution with two degrees of freedom.

THEOREM 2.5. *If $\kappa \rightarrow \infty$, then*

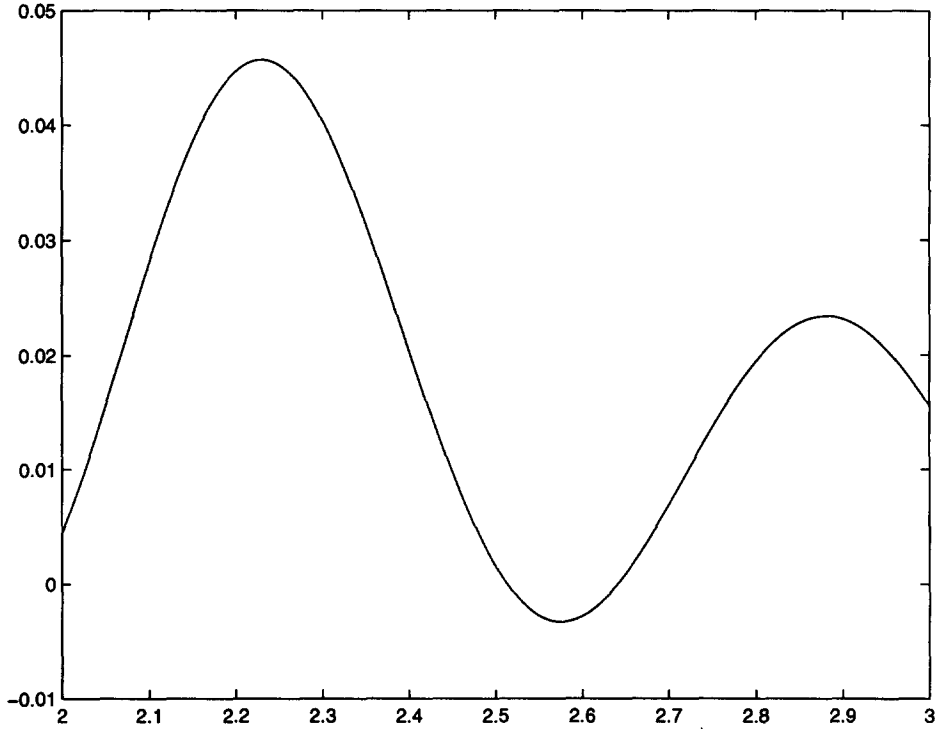
$$\lim_{\kappa \rightarrow \infty} \mathbb{P}_\kappa\left(\kappa < \hat{\kappa} + y\sqrt{\hat{\kappa}}\right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^y \exp\left\{-\frac{t^2}{2}\right\} dt,$$

i.e., $\hat{\kappa}$ has a normal distribution with variance $\mathbb{D}_\kappa^2 \hat{\kappa} \sim \kappa$.

Hence, for large κ we can apply the normal approximation, since $(\hat{\kappa} - \kappa)/\sqrt{\hat{\kappa}}$ tends to the standard normal distribution. In Table 4, the exact values of $y = \hat{\kappa}/\kappa$ are given, computed from Tables 2 and 3 (second and third row), as well as the values of y obtained by the help of the normal approximation (first row).

Table 4. The normal approximation for $\kappa = 100$.

$\lambda \setminus p$	0.100	0.050	0.025	0.010	0.001	0.999	0.990	0.975	0.950	0.900
100	1.1413	1.1832	1.2205	1.2654	1.3641	0.7385	0.7972	0.8269	0.8533	0.8847
$\frac{1}{s_2^2}$	1.1417	1.1837	1.2213	1.2664	1.3656	0.7395	0.7975	0.8269	0.8532	0.8846
$\hat{\kappa}$	1.1302	1.1716	1.2088	1.2534	1.3515	0.7330	0.7902	0.8193	0.8452	0.8762

Figure 2. The function $h_{\zeta_y}(\alpha)$ with parameters $\kappa = 0.2$, $y = 1$, $x = 10$.

3. COMPUTATIONAL TECHNIQUES

3.1. Numerical Integration

To obtain the values of the distribution functions of Ts_2^2 and ζ_y at a given point x , one has to integrate

$$h_{Ts_2^2}(\alpha) = \frac{1}{\alpha}(\mathcal{R}(\Psi_{Ts_2^2}(\alpha)) \sin(\alpha x) + \mathcal{I}(\Psi_{Ts_2^2}(\alpha))(1 - \cos(\alpha x))),$$

$$h_{\zeta_y}(\alpha) = \frac{1}{\alpha}(\mathcal{R}(\Psi_{\zeta_y}(\alpha)) \sin(\alpha x) + \mathcal{I}(\Psi_{\zeta_y}(\alpha))(1 - \cos(\alpha x))),$$

respectively, between the limits 0 and ∞ . This is done by the help of the MATLAB function `quad8` which uses an adaptive recursive Newton-Cotes 8 panel rule [23,24]. As we saw, both of the above functions are oscillating; the frequency of the oscillation depends on the value of x , the amplitude on λ and κ , respectively (see, Figures 2–5). Lemmas 2.2 and 2.3 give the idea to divide the right halfline into subintervals of length $2\pi/x$ and perform the numerical integration over these subintervals, one after the other. This method is rather similar to the one described in [25, Chapter 6]. The computational algorithm is the following.

STEP 1. Set the value of the global tolerance limit τ_0 to 10^{-8} .

STEP 2. Set the value of the local tolerance limit τ to the value of τ_0 .

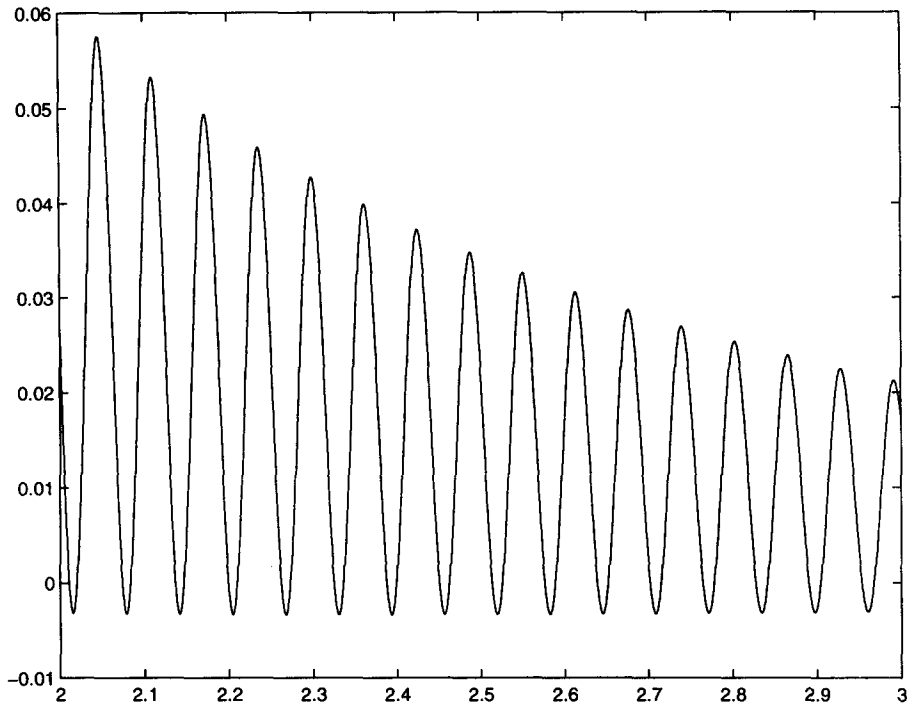


Figure 3. The function $h_{\zeta_y}(\alpha)$ with parameters $\kappa = 0.2$, $y = 1$, $x = 100$.

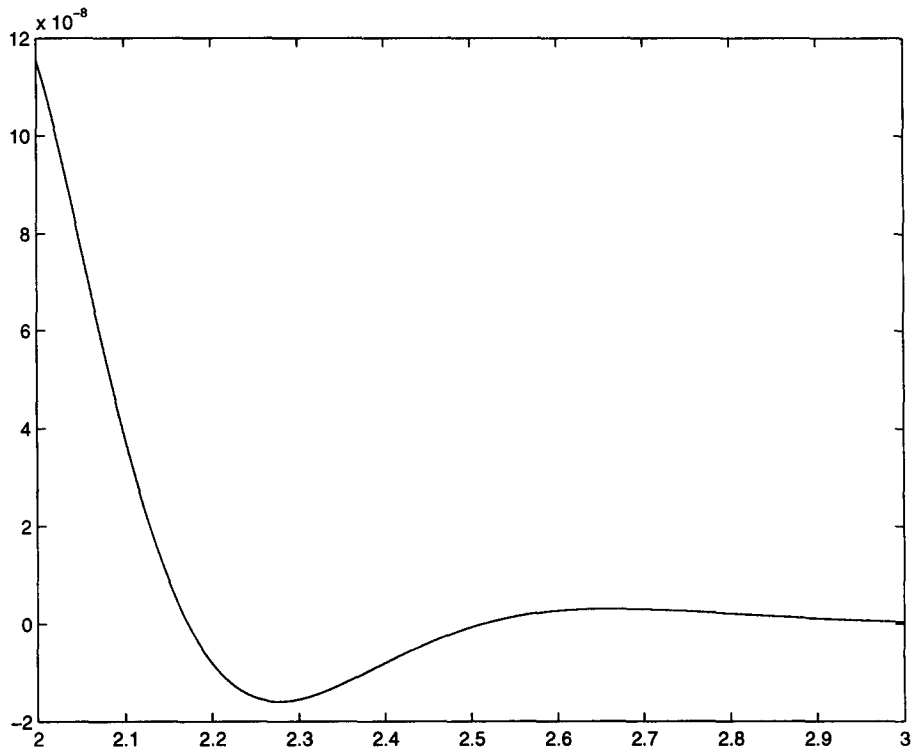


Figure 4. The function $h_{\zeta_y}(\alpha)$ with parameters $\kappa = 20$, $y = 1$, $x = 10$.

STEP 3. Perform the numerical integration over the subintervals with relative error $\tau/1000$ until the value of the integral over the last examined subinterval, e.g., $[(2\pi(\ell - 1)/x), 2\pi\ell/x]$ is less than or equal to τ . Denote I_1 the interval over $[0, 2\pi\ell/x]$.

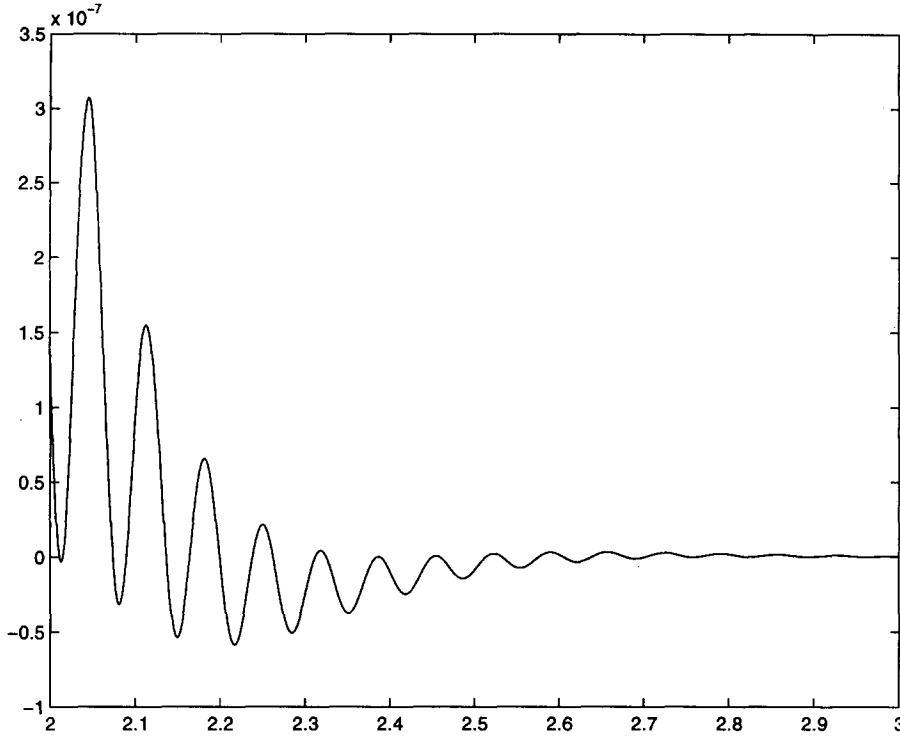


Figure 5. The function $h_{\zeta y}(\alpha)$ with parameters $\kappa = 20$, $y = 1$, $x = 100$.

STEP 4. Compute the integral over $[2\pi\ell/x, 4\pi\ell/x]$, denoted by I_2 . If $|I_2| \leq \tau$ and $2\ell \leq 10^6$, accept $I_1 + I_2$ as the integral over $[0, \infty]$. Otherwise, set τ_0 to $\tau_0/5$ and repeat Steps 3 and 4.

By Lemmas 2.2 and 2.3, both $h_{Ts_2^2}$ and $h_{\zeta y}$ die down in exponential order which justifies the above method. Usually less than 100 subintervals should be examined.

3.2. Zero Finding

To obtain the quantiles, the MATLAB function `fzero` is used. The algorithm is based on a combination of bisection, secant, and inverse quadratic interpolation methods, see [23,24]. The calculations are performed with a relative error equal to 10^{-8} . In the case of the function $h_{\zeta y}$, the starting values of the iterations are the appropriate values of the table in [3]. In the case of the function $h_{Ts_2^2}$ or at the points where there are missing values in the old table, the iteration for the calculation of the first element of a column is initialized from a reasonable-like starting guess. After that, the iteration for a given value uses the result of the previous calculation as a starting guess.

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